XVIII. *On the Formulæ investigated by Dr.* BRINKLEY *for the general Term in the Development of* LAGRANGE'S *Expression for the Summation of Series and for successive Integrations. By Sir* J. F. W. HERSCHEL, *Bart., F.R.S. &c.*

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IN the Transactions of the Royal Society for 1807, Dr. BRINKLEY has investigated the general value of the coefficient of any term in the development of the function $\left(\frac{t}{e^t-1}\right)^n$, and his result is remarkable for the mode of its expression in terms of the successive differences of the powers of zero, or of the numbers comprised in the general expression $\Delta^m 0^n$. Since that time, in my paper published in the Transactions of the Society for 1815, "On the Development of Exponential Functions," I have exhibited other, and much more simple as well as more easily calculable expressions for the same coefficient, by means of the same useful and valuable differences, and in that and other subsequent memoirs, have extended their application to a variety of interesting inquiries in the theory of differences and series. It is singular, however, that up to the present time it has never been shown that the formulæ of Dr. BRINKLEY, and my own, though affording in all cases coincident numerical results, are analytically reconcileable with each other; nor indeed is it at all easy to see either from the course of his investigation, which turns upon an intricate application of the combinatory analysis, or from the nature of the formula itself, how it is possible to pass from the one form of expression to the other so as to show their identity. This is what I now propose.

Referring to my "Collection of Examples in the Calculus of Finite Differences*," will be found the following relation, which enables us to pass from the differences of any one power of zero, as 0^x, to those of any other, as 0^{x+n}, viz.—

$$\{\log(1+\Delta)\}^n . f(\Delta)0^x = x(x-1)\dots(x-n+1) . f(\Delta)0^{x-n},$$

or changing x into $x+n$,

$$\{\log(1+\Delta)\}^n . f(\Delta)0^{x+n} = (x+1)(x+2)\dots(x+n) . f(\Delta)0^x.$$

As this equation is general so long as negative indices of Δ do not occur, we may change $f(\Delta)$ into $\frac{f(\Delta)}{(\log(1+\Delta))^n}$, and it becomes

$$f(\Delta)0^{x+n} = (x+1)\dots(x+n) . \frac{f(\Delta)}{\{\log(1+\Delta)\}^n} 0^x,$$

provided always that $\frac{f(\Delta)}{\Delta^n}$ contains no negative powers of Δ. In this for $f(\Delta)$ sub-

* Camb. 1820, p. 70. Ex. 6.

stitute Δ^{i+n}, and we have

$$\Delta^{n+i}\,0^{x+n}=\frac{[x+n]}{[x]}\,\Delta^{i}\left\{\frac{\Delta}{\log(1+\Delta)}\right\}^{n}0^{x}=\frac{[x+n]}{[x]}\,\Delta^{i}\,\nabla^{n}\,0^{x};$$

denoting by $[x]$ the continued product $1.2\ldots.x$, by $[x+n]$, $1.2.3\ldots(x+n)$, and by ∇ the combination of operations expressed by $\frac{\Delta}{\log(1+\Delta)}$, that is to say, making

$$\nabla 0^{x}=\left(1-\frac{\Delta}{2}+\frac{\Delta^{3}}{3}-\frac{\Delta^{4}}{4}+\&c.\right)^{-1}$$
$$=1+\frac{\Delta}{2}-\frac{\Delta^{2}}{12}+\frac{\Delta^{3}}{24}-\&c.;$$

so that if we put $i=0$, we shall also have

$$\Delta^{n}0^{x+n}=\frac{[x+n]}{[x]}\,\nabla^{n}0^{x},\quad \text{and}\ \frac{\nabla^{n}0^{x+n}}{[x+n]}=\frac{\nabla^{n}0^{x}}{[x]}\ .\ .\ .\ .\ .\ .\ \text{(A.)}$$

This premised, putting A_x for the coefficient of t^x in $\left(\frac{t}{e^t-1}\right)^n$, Dr. BRINKLEY'S general formula for A_x* is as follows:—

$$A_x=-n.\frac{\overline{n+2}.\overline{n+3}\ldots.\overline{n+x}}{1.2\ldots..(x-1)}\cdot\frac{1}{1}\cdot\frac{\Delta 0^{x+1}}{1.2\ldots(x+1)}+n.\overline{n+1}.\frac{\overline{n+3}\ldots.\overline{n+x}}{1.2\ldots.(x-2)}\cdot\frac{1}{1.2}\cdot\frac{\Delta^{2}0^{x+2}}{1.2\ldots..(x+2)}-\&c.,$$

or in our abbreviated notation,

$$A_x=-\frac{1}{1.(n+1)}\cdot\frac{[x+n]}{[n-1].[x-1]}\cdot\frac{\Delta 0^{x+1}}{[x+1]}+\frac{1}{1.2.(n+2)}\cdot\frac{[x+n]}{[n-1][x-2]}\cdot\frac{\Delta^{2}0^{x+2}}{[x+2]}-\&c.,$$

which, in consequence of the equation (A.), will be transformed to

$$A_x=-\frac{1}{[n-1].[x]}\left\{\frac{[x+n]}{[x-1]}\cdot\frac{1}{[n+1]}.2.3\ldots n\nabla-\frac{[x+n]}{[x-2]}\cdot\frac{1}{[n+2]}.3.4\ldots.(n+1)\nabla^{2}-\&c.\right\}0^{x}.\ \ \text{(B.)}$$

Now we have

$$(1-\nabla)^{x+n}=1-\frac{x+n}{1}\nabla+\frac{(x+n)(x+n-1)}{1.2}\nabla^{2},\ \&c.;$$

and therefore, putting S_n for the sum of the n first terms of this series,

$$(1-\nabla)^{x+n}-S_{n+1}=(-1)^{n+1}\left\{\frac{[x+n]}{[x-1]}\cdot\frac{1}{[n+1]}\nabla^{n+1}-\frac{[x+n]}{[x-2]}\cdot\frac{1}{[n+2]}\cdot\nabla^{n+2}+\&c.\right\},$$

and consequently

$$\left(\frac{d}{d\nabla}\right)^{n-1}\cdot\left\{\frac{(1-\nabla)^{x+n}-S_{n+1}}{\nabla}\right\}=(-1)^{n+1}\left\{\frac{[x+n]}{[x-1]}\cdot\frac{2.3\ldots n}{[n+1]}\nabla-\&c.\right\},$$

whence it appears that

$$A_x=(-1)^{n}\cdot\frac{1}{[n-1].[x]}\cdot\left(\frac{d}{d\nabla}\right)^{n-1}\left\{\frac{(1-\nabla)^{x+n}-S_{n+1}}{\nabla}\right\}0^{x}.$$

Dr. BRINKLEY'S expression is therefore now divested of its form in which successive different powers of 0^x occur, and reduced to one in which, according to the spirit of the algorithm adopted in my system, successive powers of Δ, or of some functions of Δ

* Philosophical Transactions, 1807, p. 125.

(as ∇) only occur, applied all to one and the same power, 0^x; and it only remains to develope the function of ∇ we have arrived at in powers of Δ, bearing in mind that $\nabla=1+\frac{\Delta}{2}-\frac{\Delta^3}{3}+$ &c., or $1-\nabla=\Delta\left\{-\frac{1}{2}+\frac{\Delta}{3}-\frac{\Delta^2}{4}+\text{ \&c.}\right\}$.

Now as regards the term $\frac{(1-\nabla)^{x+n}}{\nabla}$, it is obvious that it will take when developed the form $\Delta^{x+n}.\{\alpha+\beta\Delta+\gamma\Delta^2+\text{ \&c.}\}$, and therefore, that when differentiated $n-1$ times successively with respect to Δ, it will have the form $\Delta^{x+1}\{\alpha'+\beta'\Delta+\gamma'\Delta^2+\text{ \&c.}\}$, so that, when applied as an operative symbol to 0^x, the result will be $=0$. This term then may be simply struck out, and we have only to consider the development in Δ of $\left(\frac{d}{d\nabla}\right)^{n-1}.\frac{-S_{n+1}}{\nabla}.0^x$.

Now this is equal to

$$-\left(\frac{d}{d\nabla}\right)^{n-1}.\left\{\frac{1}{\nabla}-(x+n)+\frac{(x+n)(x+n-1)}{1.2}\nabla-\ldots\pm\frac{[x+n]}{[n].[x]}\nabla^{n-1}\right\}0^x$$

$$=-\left\{\left(\frac{d}{d\nabla}\right)^{n-1}.\frac{1}{\nabla}-0+0+0\ldots\ldots\qquad\pm\frac{[x+n]}{n.[x]}\right\}0^x,$$

all the terms of which vanish except the first, x being >-1, and the whole reduces itself to $-\left(\frac{d}{d\nabla}\right)^{n-1}.\frac{1}{\nabla}0^x$, or to $(-1)^n.\frac{[n-1]}{\nabla^n}0^x$. And we therefore obtain, finally,

$$A_x=\frac{1}{[x]}.\frac{1}{\nabla^n}0^x=\frac{1}{1.2\ldots\ldots x}.\left\{\frac{\log(1+\Delta)}{\Delta}\right\}^n.0^x,$$

which is identical with that given in my paper above mentioned, equation (6), or in the "Examples," p. 82.

J. F. W. HERSCHEL.

Collingwood, April 20, 1860.

Note by A. CAYLEY, *M.A., F.R.S.*

The above formula (B.), substituting therein for A_x the value $\frac{1}{[x]}\nabla^{-n}0^x$, becomes

$$\nabla^{-n}0^x=-\frac{1}{[n-1]}\left\{\frac{[x+n]}{[x-1]}\,\frac{1}{[n+1]}2.3..n\,\nabla-\frac{[x+n]}{[x-2]}\,\frac{1}{[n+2]}3.4\ldots\overline{n+1}\,\nabla^2-\text{ \&c.}\right\}0^x;$$

or, as this may be written,

$$\nabla^{-n}0^x=\frac{[x+n]}{[n-1]}\left\{-\frac{1}{[1][x-1](n+1)}\nabla+\frac{1}{[2][x-2](n+2)}\nabla^2-\text{ \&c.}\right\}0^x;$$

or, inserting a first term which vanishes except in the case $x=0$, and which is required in order that the formula may hold good for this particular value,

$$\nabla^{-n}0^x=\frac{[x+n]}{[n-1]}\left\{\frac{1}{[0][x]n}\nabla^0-\frac{1}{[1][x-1](n+1)}\nabla+\frac{1}{[2][x-2](n+2)}\nabla^2-\text{ \&c.}\right\}0^x;$$

where the series on the right-hand side need only be continued up to the term containing $\nabla^x 0^x$, since the subsequent terms vanish.

Now $\nabla^{-n}0^x$, or $\left(\frac{\Delta}{\log(1+\Delta)}\right)^{-n}0^x$, is equal to $[x]\times$coef. t^x in $\left(\frac{e^t-1}{t}\right)^{-n}$, and so ∇^q0^x, or $\left(\frac{\Delta}{\log(1+\Delta)}\right)^q0^x$, is equal to $[x]\times$coef. t^x in $\left(\frac{e^t-1}{t}\right)^q$.

Hence, putting $\mathrm{R}=\frac{e^t-1}{t}$, the last-mentioned formula will be true if, as regards the term which contains t^x, we have

$$\mathrm{R}^{-n}=\frac{[x+n]}{[n-1]}\left\{\frac{1}{[0]\,[x]n}\,\mathrm{R}^0-\frac{1}{[1]\,[x-1](n+1)}\,\mathrm{R}^1+\frac{1}{[2]\,[x-2](n+2)}\,\mathrm{R}^2-\&c.\right\},$$

the series on the right-hand side being continued up to the term in R^x. This formula is, in fact, true if R, instead of being restricted to denote $\frac{e^t-1}{t}$, denotes any function whatever of the form $1+bt+ct^2+\&c.$, and it is true not only for the term in t^x, but for all the powers of t not higher than t^x. And, moreover, R^{-n} may denote any positive or negative integral or fractional power of R. In fact, the formula (assuming for a moment the truth of it) shows that the expansion of any power whatever of a series of the form in question, can be obtained by means of the expansions of the successive positive integer powers of the same series: the existence of such a formula (at least for negative powers) was indicated by EISENSTEIN, *Crelle*, t. xxxix. p. 181 (1850), and the formula itself, in a slightly different form, was obtained in a very simple manner by Professor SYLVESTER in his paper, "Development of an idea of EISENSTEIN," Quart. Math. Journ. t. i. p. 199 (1855); the demonstration was in fact as follows, viz. writing

$$\mathrm{R}^n=(1+\overline{\mathrm{R}-1})^n=1+\frac{n}{1}(\mathrm{R}-1)+\frac{n.n-1}{1.2}(\mathrm{R}-1)^2+\&c.:$$

if we attend only to the terms involving powers of t not higher than t^x, the series on the right-hand side need only be continued up to the term involving $(\mathrm{R}-1)^x$, and the right side being thus converted into a rational and integral function of R, it may be developed in a series of powers of R (the highest power being of course R^x), and the coefficients of the several powers are finite series which admit of summation; this gives the required formula. But there is an easier method; the process shows that the series on the right-hand side, continued as above up to the term involving t^x, is, as regards n, a rational and integral function of the degree x; and by LAGRANGE's interpolation formula, any rational and integral function of n of the degree x, can be at once expressed in terms of the values corresponding to $(x+1)$ particular values of n. The investigation will be as follows:—Let R denote a series of the form $1+bt+ct^2+\&c.$, and let R^n denote the development of the nth power of R, continued up to the term containing t^x, the terms involving higher powers of t being rejected. R^0, R^1, R^2, &c., and generally R^s, will in like manner denote the developments of these powers up to the term involving t^x, or what is the same thing, they will be the values of R^n, corresponding to $n=0, 1, 2,..s$. By what precedes R^n is a rational and integral function of n of the degree x, and it can therefore be expressed in terms of the values R^0, R^1, R^2, ... R^x, which correspond to $n=0, 1, 2, .\, x$.

Let s have any one of the last-mentioned values, then the expression

$$\frac{n.n-1.n-2\ldots n-x}{n-s}\;\frac{1}{s.s-1\ldots 2.1.-1.-2\ldots-(x-s)},$$

which, as regards n, is a rational and integral function of the degree x (the factor $n-s$, which occurs in the numerator and in the denominator being of course omitted), vanishes for each of the values $n=0, 1, 2, \ldots x$, except only for the value $n=s$, in which case it becomes equal to unity. The required formula is thus seen to be

$$\mathrm{R}^n=\Sigma\left\{\frac{n.n-1.n-2\ldots n-x}{n-s}\;\frac{1}{s.s-1\ldots 2.1.-1.-2\ldots-(x-s)}\,\mathrm{R}^s\right\},$$

where the summation extends to the several values $s=0, 1, 2\ldots x$; or, what is the same thing, it is

$$\mathrm{R}^n=\Sigma\left\{\frac{n.n-1.n-2..n-x}{n-s}\;\frac{(-)^{x-s}}{1.2\ldots s.1.2\ldots(x-s)}\,\mathrm{R}^s\right\};$$

or changing the sign of n, it is

$$\mathrm{R}^{-n}=\Sigma\left\{\frac{n.n+1.n+2..n+x}{n+s}\;\frac{(-)^s}{1.2\ldots s.1.2\ldots x-s}\,\mathrm{R}^s\right\},$$

or, as this may be written,

$$\mathrm{R}^{-n}=\frac{[x+n]}{[n-1]}\,\Sigma\left\{\frac{(-)^s}{[s]\,[x-s]\,(n+s)}\,\mathrm{R}^s\right\}$$

or substituting for s the values $0, 1, 2, \ldots x$, the formula is

$$\mathrm{R}^{-n}=\frac{[x+n]}{[n-1]}\left\{\frac{1}{[0]\,[x]n}\,\mathrm{R}^0-\frac{1}{[1]\,[x-1]\,(n+1)}\,\mathrm{R}^1+\frac{1}{[2]\,[x-2]\,(n+2)}\,\mathrm{R}^2-\ldots\right\}$$

continued up to the term involving R^x, which is the theorem in question.